4.2 – Subspaces

Definition: A subset W of a vector space V is called a **subspace** of V if W is itself a vector space under the addition and scalar multiplication defined on V.

Theorem 4.2.1 Subspace Test

If W is a nonempty set of vectors in a vector space V, then W is a subspace of V if and only if the following conditions are satisfied.

a) If $\mathbf{u}$ and $\mathbf{v}$ are vectors in W, then $\mathbf{u} + \mathbf{v}$ is in W.

b) If k is a scalar and $\mathbf{u}$ is a vector in W, then $k\mathbf{u}$ is in W.

Example: (3) Use the Subspace Test to determine which of the sets are subspaces of M_{nn}.

a. The set of all diagonal $n \times n$ matrices.

b. The set of all $n \times n$ matrices A such that $\det(A) = 0$.

c. The set of all $n \times n$ matrices A such that $\text{tr}(A) = 0$.

d. The set of all symmetric $n \times n$ matrices.

Example: (6) Use the Subspace Test to determine which of the sets are subspaces of P_3.

a. All polynomials of the form $a_0 + a_1 x + a_2 x^2 + a_3 x^3$ in which a_0, a_1, a_2, and a_3 are rational numbers.

b. All polynomials of the form $a_0 + a_1 x$ in which a_0 and a_1 are real numbers.

Example: (11) Use the Subspace Test to determine which of the sets are subspaces of M_{22}.

a. All matrices of the form $\begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix}$.

b. All matrices of the form $\begin{bmatrix} a & 1 \\ b & 1 \end{bmatrix}$.

c. All 2×2 matrices A such that $A \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$.

Example: (14) Use the Subspace Test to determine which of the sets are subspaces of R^4.

a. All vectors $\mathbf{x}$ in R^4 such that $A\mathbf{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, where $A = \begin{bmatrix} 0 & -1 & 0 & 2 \\ -1 & 1 & 0 & 1 \end{bmatrix}$

b. All vectors $\mathbf{x}$ in R^4 such that $A\mathbf{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, where A is as in part (a).

Theorem 4.2.3 The solution set of a homogeneous system $A\mathbf{x} = \mathbf{0}$ of m equations in n unknowns is a subspace of R^n.

Definition: The solution set of a homogeneous system in n unknowns is a subspace of R^n, called the **solution space** of the system.

Definition: Let $T_A : R^n \to R^m$ be multiplication by the coefficient matrix A. The solution space of $A\mathbf{x} = \mathbf{0}$ is the set of vectors in R^n that T_A maps into the zero vector in R^m. This set is called the **kernel** of the transformation.

Theorem 4.2.4 If A is an $m \times n$ matrix, then the kernel of the matrix transformation $T_A : R^n \to R^m$ is a subspace of R^n.

Theorem 4.2.2 If $W_1, W_2, \ldots, W_r$ are subspaces of a vector space V, then the intersection of these subspaces is also a subspace of V.

Examples of Subspaces
$\{\mathbf{0}\}$ is a subspace of every vector space V
Any vector space V is a subspace of itself

Subspace of R^2
Lines through the origin

Subspaces of R^3
Lines through the origin
Planes through the origin

The solution space of a homogeneous system in n unknowns is a subspace of R^n

Subspaces of M_{nn}
Symmetric matrices
Triangular matrices
Diagonal matrices

Subspaces of $F(-\infty, \infty)$ (The following is actually a sequence of nested subspaces)
$C(-\infty, \infty)$, the set of functions continuous on R
$C^1(-\infty, \infty)$, the set of functions with continuous first-order derivatives on R
$C^n(-\infty, \infty)$, the set of functions with continuous n^{th}-order derivatives on R
$C^\infty(-\infty, \infty)$, the set of functions with derivatives of all orders on R
P_∞, the set of polynomials
P_n, polynomials of degree $\leq n$